LECTURES IN IOANINNA

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ABSTRACT. One of the main themes of this conference addresses invariants, as well as their connection to endomorphisms of functors. These endomorphisms yield useful techniques for the analysis of certain natural problems in the subject.

At this point there is a dichotomy: Smith's lectures focus on functors which reflect "abelian properties" of an object in Algebraic Topology while Cohen's lectures focus on functors which reflect "non-abelian properties" in Algebraic Topology. Part of the role, and applications of these structures for classical homotopy groups will be addressed in these lectures. The four topics considered are as follows:

1. Splittings of spaces,
2. Endomorphisms of tensor algebras, and self-maps of loop spaces,
3. Braid groups, and homotopy groups of the 2-sphere, and
4. Cohomology of symmetric groups, and other groups.

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1. Splittings of spaces

The basic objects of study here are loop spaces, and suspension spaces. One goal is to obtain information about the homotopy groups, and homology groups of these spaces. These constructions also provide a natural continuation of the themes in Professor Smith's lectures in which he discusses classifying spaces.

For example, Milnor [20] showed that the loop space of a path-connected simplicial complex X has the homotopy type of a topological group G, and where BG has the homotopy type of X. This construction will be used below.

Principle: Many useful spaces are homotopy equivalent to a classifying space. This feature has informative consequences as illustrated below.

Theorem 1.1. *Let X be a topological space which is a connected CW-complex. Then there exists a topological group G such that X is homotopy equivalent to BG.*

Next consider the path loop fibration

$$\Omega X \to PX \to X$$

where PX denotes the path-space, the space of continuous functions $\{f : [0,1] \to X | f(0) = *\}$, and ΩX is the subspace of PX given by

$$\{f : [0,1] \to X | f(0) = * = f(1)\}.$$

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Classically, there are isomorphisms

$$\pi_i(X) \to \pi_{i-1}(\Omega X).$$

Thus the study of the homotopy groups of a simply-connected space reduces to those for its' loop space. The point of view of these lectures is to regard X as a classifying space, and to learn properties about X from the loop space of X. Many of these properties are described in [30].

There are sometimes useful, and informative features of ΩX which then inform on X. It will be seen below that certain choices of invariant elements under the action of a symmetric group yield additional information about ΩX. Thus to continue a second theme in Professor Smith's lectures of invariant elements arising from natural actions of groups, the symmetric groups are used to give decompositions of certain loop spaces which arise from such invariants.

The first example in this context arises from the classical Hopf fibrations: There are homotopy equivalences

1. $\Omega S^2 \to S^1 \times \Omega S^3$,
2. $\Omega S^4 \to S^3 \times \Omega S^7$, *and*
3. $\Omega S^8 \to S^7 \times \Omega S^{15}$.

These decompositions represent reformulations of the classical Hopf invariant one problem, and are the the only cases for which such product decompositions for loop spaces of spheres exist. That these are the only spheres S^{n+1} such that ΩS^{n+1} is homotopy equivalent to $S^n \times \Omega S^{2n+1}$ is equivalent to the classical result on the non-existence of elements of Hopf invariant one.

However, after localization away from 2, there are homotopy equivalences

$$\Omega S^{2n} \to S^{2n-1} \times \Omega S^{4n-1}$$

given by classical results due to Serre. These decompositions will be reformulated below in terms of invariants of actions for certain symmetric groups. A general context together with applications to other natural spaces are given as well.

One method of proof here is as follows. Start with a principle G-bundle with a cross-section. It is a classical fact that such bundles are trivial. This method provides a process for showing that a loop space of X is sometimes homotopy equivalent to a product.

Proposition 1.2. *Let $E \to B$ be a fibration with homotopy theoretic fibre F. Assume that the natural map $i : F \to$ E admits a cross-section up to homotopy (thus there is a map $\sigma : E \to F$ such that $i \circ \sigma$ is homotopic to the identity). Then there is a homotopy equivalence*

$$F \to E \times \Omega B.$$

As an example, consider the natural homomorphisms $S^1 \to S^3$ to obtain a fibration $BS^1 \to BS^3$ with fibre S^2. "Backing up" this fibration, there is an induced fibration $\Omega S^2 \to S^1$ with homotopy theoretic fibre ΩS^3. Since there is a section for this last fibration, the product decomposition $\Omega S^2 \to S^1 \times \Omega S^3$ follows at once. This gives the product decomposition for ΩS^2 listed above. The other cases are similar.

Other important constructions which fit in this framework are the Whitehead product, and the Samelson product. Here consider a topological group G. The commutator map

$$[-,-] : G \times G \to G$$

is gotten by sending the ordered pair (a, b) to the commutator $[a, b] = a^{-1}b^{-1}ab$. Notice that if either a or b is equal to 1, then the commutator $[a, b]$ is equal to 1. There is an induced map

$$S : G \wedge G \to G$$

where $G \wedge H$ denotes the quotient $G \times H/(G \times \{1\} \cup \{1\} \times H)$.

Notice that $S^k \wedge S^n = S^{n+k}$. There is an induced bilinear map for all $i, j \geq 1$ on the level of homotopy groups

$$S_* : \pi_i G \otimes \pi_j G \to \pi_{j+k} G.$$

Regarding ΩX as a topological group G, there are analogous pairings induced on the level of homotopy groups for any simply-connected CW complex X. These pairings satisfy the (graded) antisymmetry law for a Lie bracket if the prime 2 is a unit, as well as the (graded) Jacobi identity if 3 is a unit. In the case of graded Lie algebras over $\mathbb{F}_2$, the element $[x, x]$ is required to be zero, and over $\mathbb{F}_3$, the element $[[x, x], x]$ is required to be zero. These last two properties fail in general for the Samelson product S_* without additional assumptions. In case x is of even degree, it is the case that $[x, x]$ is sometimes non-zero, and is of order 2. In case x is of odd degree, it is the case that $[[x, x], x]$ is sometimes non-zero, and is of order 3.

This pairing S_* is known as the Samelson product [26]. This pairing is related to the classical Whitehead product W by adjointness up to a sign, and gives the following commutative diagram where $\alpha : \pi_{q+1} X \to \pi_p \Omega X$ is the adjoint yielding a natural isomorphism, and the pairing $[x, y]$ in the homology of ΩX is given by $x \otimes y - (-1)^{degree(x)degree(y)} y \otimes x$ [6], page 215:

$$\begin{array}{ccc}
\pi_{p+1} X \otimes \pi_{q+1} X & \xrightarrow{W} & \pi_{p+q+1} X \\
\downarrow{\scriptstyle \alpha \otimes \alpha} & & \downarrow{\scriptstyle \alpha} \\
\pi_p \Omega X \otimes \pi_q \Omega X & \xrightarrow{S_*} & \pi_{p+q} \Omega X \\
\downarrow & & \downarrow \\
H_p \Omega X \otimes H_q \Omega X & \xrightarrow{[-,-]} & H_{p+q} \Omega X
\end{array}$$

These constructions give rise to product decompositions of loop spaces in much the same way that elements of Hopf invariant one give rise to product decompositions above. This feature will be seen after the next example for which a product decomposition of a loop space arises from work of T. Ganea, and P. Hilton, and others.

Here consider the inclusion of the wedge $X \vee Y$ in the product $X \times Y$ with homotopy theoretic fibre F. Let $X^{(k)}$ denote the k-fold smash product where $X \wedge Y$ denotes $X \times Y/X \vee Y$. The following is a theorem of T. Ganea [14].

Theorem 1.3. *Let X, and Y be connected CW complexes, with the homotopy theoretic fibre of the natural inclusion $X \vee Y$ in $X \times Y$ denoted by F. Then*

1. *F is homotopy equivalent to the $\Sigma(\Omega X \wedge \Omega Y)$.*
2. *There is a homotopy equivalence*

$$\Omega(X) \times \Omega(Y) \times \Omega\Sigma(\Omega X \wedge \Omega Y) \rightarrow \Omega(X \vee Y).$$

3. *Furthermore, the choice of map $\Omega\Sigma(\Omega X \wedge \Omega Y) \rightarrow \Omega(X \vee Y)$ is the canonical multiplicative extension of the composition of the map*

$$\Omega(i) \wedge \Omega(j) : \Omega(X) \wedge \Omega(Y) \rightarrow \Omega(X \vee Y) \wedge \Omega(X \vee Y),$$

with the commutator map

$$S : \Omega(X \vee Y) \wedge \Omega(X \vee Y) \rightarrow \Omega(X \vee Y)$$

where $i : X \rightarrow X \vee Y$, and $j : Y \rightarrow X \vee Y$ are given by the natural inclusions.

Thus for example, if X and Y are $\mathbb{CP}^\infty$, then X, and Y have precisely one non-vanishing homotopy group. Since $\Omega \mathbb{C}P^\infty$ is homotopy equivalent to S^1, the theorem implies a homotopy equivalence

$$\Omega(\mathbb{CP}^\infty \vee \mathbb{C}P^\infty) \rightarrow S^1 \times S^1 \times \Omega S^3.$$

Thus the homotopy groups of $\mathbb{CP}^\infty \vee \mathbb{CP}^\infty$ are those of the 3-sphere plus 2 other copies of the integers (in degree 2). However, the spaces $\mathbb{CP}^\infty \vee \mathbb{CP}^\infty$, and $S^3 \times \mathbb{CP}^\infty \times \mathbb{CP}^\infty$ are not homotopy equivalent.

The next theorem is the classical Hilton-Milnor theorem in which the notation $X^{(k)}$ is used for the k-fold smash product as above.

Theorem 1.4. *Let X, and Y be connected CW complexes. Then there is a homotopy equivalence*

$$\Omega\Sigma(X \vee \Sigma Y) \rightarrow \Omega\Sigma(X) \times \Omega\Sigma(Y) \times \Omega\Sigma(\vee_{i,j\geq 1} X^{(i)} \wedge Y^{(j)}).$$

How do these "fit" with invariants ? How do they arise in some further way ? What are these good for ? Some of these questions will be addressed next. Two examples for which product decompositions have proven to be useful are listed next.

1. Product decompositions for the loop space of a $mod-p^r$ Moore space for p prime have useful applications. These decompositions impinge on the structure of the homotopy groups of spheres as well as other finite complexes.
2. More general splittings will be illustrated where spaces are localized at a fixed prime. For example, consider spaces X which are (1) double suspensions, and (2) their homology groups are non-trivial, and entirely torsion. Then the loop space of X admits a product decomposition with infinitely many non-trivial factors. These decompositions then directly give non-trivial elements in homotopy groups.

In what follows below, it will be assumed that the reduced homology of the spaces below are entirely torsion.

There are natural self-maps of $X^{(k)}$ given by elements in the symmetric group on k letters Σ_k. Thus, there is an "action" of the integral group ring $\mathbb{Z}[\Sigma_k]$on $\Sigma X^{(k)}$ given by adding via the suspension coordinate. These self-maps have been used widely to give decompositions of $\Sigma X^{(k)}$. Some examples are listed below.

Example 1.5. Example: Let k = 2, and let β_2 denote the element in the group ring given by $1-(1,2)$ where $(1,2)$ is the transposition which interchanges 1, and 2. Then a direct computation gives $(\beta_2)^2 = 2\beta_2$.

Furthermore, if 2 is a unit in the reduced homology of X, then the elements β_2, and $2-\beta_2$ give an orthogonal decomposition of the homology of $\Sigma X^{(2)}$. In addition, if 2 is unit in the reduced integer homology of $\Sigma X^{(2)}$, then there is a homotopy equivalence

$$\Sigma X^{(2)} \rightarrow L_2 \vee M_2$$

where L_2 denotes the homotopy direct limit of β_2, and M_2 denotes the homotopy direct limit of $2-\beta_2$. This example is expanded below.

The Dynkin-Specht-Wever elements β_n are elements in the integral group ring of the symmetric group $\mathbb{Z}[\Sigma_n]$, which can be defined as follows: Regard the n-fold tensor product $V^{\otimes n}$ as a module over $\mathbb{Z}[\Sigma_n]$. Then β_n in $\mathbb{Z}[\Sigma_n]$ is obtained by the linear transformation which sends $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ to the element $[[\cdot[v_1, v_2]v_3]\cdots]v_n]$ where the bracket $[x,y]$ means $x \otimes y - (-1)^{deg(x)deg(y)} y \otimes x$.

For simplicity, assume that a space Y is a suspension ΣX. There are induced self-maps $\beta_n, n-\beta_n : (\Sigma X)^{(n)} \rightarrow (\Sigma X)^{(n)}$. Let $L_n(\Sigma X)$ denote the homotopy direct limit of β_n and $M_n(\Sigma X)$ the homotopy direct limit of $n-\beta_n$.

Proposition 1.6.
1. *The formula $\beta_n \circ \beta_n = n\beta_n$ holds in homology for the self-maps $\beta_n : (\Sigma X)^{(n)} \rightarrow (\Sigma X)^{(n)}$.*
2. *If n is a unit in the reduced homology of X, then there is map which induces a homology isomorphism $(\Sigma X)^{(n)} \rightarrow L_n(\Sigma X) \vee M_n(\Sigma X)$.*

The proof is that singular homology commutes with the direct limit construction here, and that the maps are isomorphisms in homology with any field coefficients. The proof is a special case of what follows below. These constructions were given in work of the author, and J. Wu, and have been developed further in recent work of P. Selick, and J. Wu.

Namely, let $g : V \rightarrow V$ be an idempotent self-map of a vector space V. Thus $g^2 = g$, and so g, and $1-g$ give an orthogonal idempotents of V, and there is an isomorphism of vector spaces $V \rightarrow gV \oplus (1-g)V$. Notice that there is an isomorphism $gV \rightarrow \operatorname{inj\,lim}_g V$. This proof has a topological analogue.

Proposition 1.7. *Let $f : \Sigma X \rightarrow \Sigma X$ be any map which is idempotent on the level of reduced homology groups. Then there is a map*

$$\Sigma X \rightarrow A \vee B$$

which induces an isomorphism on homology where

1. $A = \operatorname{inj\,lim}_f \Sigma X$, *and*
2. $B = \operatorname{inj\,lim}_{1-f} \Sigma X$

Thus if X has the homotopy type of a CW-complex, the map Θ is a homotopy equivalence.

This basic idea has been exploited in many beautiful ways by G. Cooke, N. Kuhn, S. Mitchell, G. Nishida, N. Ray, J. Smith, L. Smith, and R. Wood as well as many others.

The main point here is that invariants in linear algebra give topological information by exhibiting coalgebra decompositions of tensor algebras which can then be realized by topological spaces. Using this principle, the following theorem was proven in [8].

Theorem 1.8. *Fix a prime p, and assume that n is unit in the reduced mod-p homology of a CW-complex X. Then there is a homotopy equivalence*

$$\Omega\Sigma^2 X \to \Omega\Sigma^2 L_n(X) \times B(X)$$

for some choice of space $B(X)$. In addition, the mod-p homology of $L_n(X)$ is isomorphic to the module of Lie elements of tensor weight n in the tensor algebra $T[V]$ where V is the reduced mod-p homology of ΣX. Thus if the reduced mod-p of X has at least two linearly independent elements, then $L_n(X)$ has non-trivial homology for every n prime to p.

A specific example of the above theorem where X is a 2-cell complex given by a mod-2 Moore space is described below. A sketch of the proof of this theorem is given before this example as follows:

1. The Samelson product yields a map $Y^{(n)} \to \Omega\Sigma Y$.
2. Specialize to $Y = \Sigma X$ and appeal to Proposition 1.6 to obtain a map $L_n(\Sigma X) \to \Omega\Sigma Y$ with canonical multiplicative extension
$$\Omega\Sigma L_n(\Sigma X) \to \Omega\Sigma Y.$$
3. The Hopf invariant construction discussed in the next section is a map
$$\Omega\Sigma Y \to \Omega\Sigma Y^{(n)}.$$
Again, let $Y = \Sigma X$, and use Proposition 1.6 to project $\Omega\Sigma Y^{(n)} \to \Omega\Sigma L_n(\Sigma X)$.
4. The composite
$$\Omega\Sigma L_n(\Sigma X) \to \Omega\Sigma^2 X \to \Omega\Sigma L_n(\Sigma X)$$
induces a homology isomorphism by a direct (messy) computation. Alternatively, the composite can be shown to be homotopic to a loop map, and the computation is then direct.

Let $P^{n+1}(2)$ denote $\Sigma^{n-1}\mathbb{RP}^2$ where $\mathbb{RP}^2$ is the real projective plane. The next theorem follows by substituting $n = 3$ or 5 in the previous theorem where $P^n(2)$ denotes the (n-1)-sphere with an n cell attached by a degree 2 map. Thus there is a homotopy equivalence $\Sigma^{n-2}\mathbb{RP}^2 \to P^n(2)$.

Theorem 1.9. *Assume that $n > 2$. Then there are homotopy equivalences as follows:*

$$\Omega P^{n+1}(2) \simeq \begin{cases} \Omega P^{4\mu(5m-2,k)+2}(2) \times X(n+1) & \text{if } n+1 = 4m, \\ \Omega P^{4\mu(15m-2,k)+2}(2) \times Y(n+1) & \text{if } n+1 = 4m+1, \\ \Omega P^{4\mu(m,k)+2}(2) \times Z(n+1) & \text{if } n+1 = 4m+2, \\ \Omega P^{4\mu(3m+1,k)+2}(2) \times W(n+1) & \text{if } n+1 = 4m+3 \end{cases}$$

for all $k \geq 1$, where μ is defined by

$$\mu(n,k) = 9^{k-1}n + \sum_{j=0}^{k-2} 9^j.$$

It was proven by J. Mukai [23] or in [8] that if $n \geq 4$, and n is odd, then $\pi_{4n-2}P^n(2)$ contains $\mathbb{Z}/8\mathbb{Z}$.

Proposition 1.10. *If $n > 3$, then there are infinitely many elements of order 8 in the homotopy groups of $P^n(2)$.*

There are two outstanding conjectures in this subject:

1. Barratt's finite exponent conjecture: Assume that the suspension order of the identity for $\Sigma^2 X$ is p^r. Then p^{r+1} annihilates the homotopy groups of $\Sigma^2 X$.
2. Moore's conjecture: Assume that X is a simply-connected finite complex which has finitely many non-zero rational homotopy groups $\pi_i X \otimes \mathbb{Q}$. Then for any fixed prime p, the p-torsion in the homotopy groups of X have a bounded exponent for all i (depending on p).

The work above was directed toward considering these questions for mod-2 Moore spaces, and was inspired by the following two theorems which were proven much earlier using splitting techniques.

Theorem 1.11. [7] *If p is an odd prime, then p^n annihilates the p-torsion in the homotopy groups of S^{2n+1}.*

Theorem 1.12. [25] *If p is an odd prime, then p^{r+1} annihilates the homotopy groups of a simply-connected $mod - p^r$ Moore space $P^{n+1}(p^r)$.*

2. Endomorphisms of tensor algebras, and self-maps of loop spaces

In the previous lecture, certain product decompositions for loop spaces arose from natural coalgebra decompositions of the tensor algebra. This theme will be pursued here where the collection of all natural transformations with respect to certain analogous structures will be discussed.

Consider a graded free module V over the integers $\mathbb{Z}$ or a field $\mathbb{F}$. The modules V considered here will usually arise as the reduced homology groups of a path-connected space X. Thus it will be assumed that V is concentrated in degrees strictly greater than 0. Let $T[V]$ denote the tensor algebra generated by V.

Then $T[V] = \bigoplus_{n \geq 0} V^{\otimes n}$. In addition, $T[V]$ inherits the natural structure of a Hopf algebra by requiring the elements in V to be primitive, and thus $\Delta(v) = v \otimes 1 + 1 \otimes v$ for v in V where Δ denotes the coproduct. Hence, there is a natural diagonal map which is a morphism of Hopf algebras:

$$\Delta : T[V] \to T[V] \otimes T[V].$$

Notice that $T[V]$ is a functor from graded modules to graded Hopf algebras. One might ask for the natural transformations from this functor to itself which preserves the underlying structure of a coalgebra.

Part of the motivation here is that the tensor algebra $T[V]$ gives the homology of certain families of topological spaces by the following:

Theorem 2.1. *(Bott-Samelson) Let X be a topological space which is a connected CW-complex. Assume either that the integer homology is either (1) torsion free, or (2) the homology groups are taken with field coefficients $\mathbb{F}$. Let V denote the reduced homology of X. Then there is an isomorphism of algebras*

$$\Theta : T[V] \to H_*(\Omega\Sigma X).$$

If in addition, X is a suspension, then Θ is an isomorphism of Hopf algebras.

One could ask about self-maps of $\Omega\Sigma X$. The action of such a map in homology then gives a morphism of coalgebras of $T[V]$. One could ask further about the "generic self-maps", those maps which are natural for all spaces X, or all modules V.

First notice that the set of coalgebra self-maps of $T[V]$,

$$Hom^{coalg}(T[V], T[V])$$

forms a group where the product of two elements is defined as follows:

$$\begin{array}{ccc} T[V] & \xrightarrow{\Delta} & T[V] \otimes T[V] \\ \downarrow \text{id} & & f\otimes g \downarrow \\ T[V] & \longrightarrow & T[V] \otimes T[V] \\ \downarrow \text{id} & & multiply \downarrow \\ T[V] & \xrightarrow{f \cdot g} & T[V] \end{array}$$

The basic point here is that $T[V]$ admits natural self-maps which in fact correspond to self-maps of spaces. The structure of these then inform on spaces. In addition, Artin's braid group arises, and plays a significant role within classical homotopy theory.

Next consider self-maps of $T[V]$ as follows:

Definition 2.2. Let q be an integer.

1. The map $\phi_q : T[V] \to T[V]$ is given by the multiplicative map which sends each element v to qv.
2. The map $\psi_q : T[V] \to T[V]$ is given by the $q-th$ power map
3. The map $\lambda_q : T[V] \to T[V]$ is given by that map induced in homology by the composite
$$\Omega\Sigma^2 X \to \Omega\Sigma(\Sigma X)^{(n)} \to \Omega\Sigma^2 X$$
of
   (a) the q-th Hopf invariant $h_q : \Omega\Sigma^2 X \to \Omega\Sigma(\Sigma X)^{(q)}$ with
   (b) $\Omega\Sigma(\Sigma X)^{(n)} \to \Omega\Sigma^2 X$, the looping of the q-fold Whitehead product.

These maps all give natural transformations of $T[V]$. The maps $\lambda_q : T[V] \to T[V]$ are non-trivial, and intricate. Let H_∞ denote the group generated by these elements.

Theorem 2.3. *The group H_∞ is the inverse limit of a system*

$$\cdots \to H_n \to H_{n-1} \to \cdots \to H_2 \to H_1.$$

The maps $H_n \to H_{n-1}$ are non-split epimorphisms of groups with kernel given by the center $Lie(n)$ of H_n. Furthermore, $Lie(n)$ is a free abelian group of rank $(n-1)!$.

The algebraic maps above are all realized by self-maps of $\Omega\Sigma^2X$. Thus there is a group homomorphism from the free group F generated by the elements in Definition 2.2 to the group of homotopy classes of self-maps $[\Omega\Sigma^2X, \Omega\Sigma^2X]$, say

$$\phi(X) : F \to [\Omega\Sigma^2X, \Omega\Sigma^2X]$$

together with an induced homomorphism

$$\Phi : F/\cap ker\phi(X) \to [\Omega\Sigma^2X, \Omega\Sigma^2X]$$

where the intersection is over every space X.

Theorem 2.4. *The group H_∞ is isomorphic to $F/\cap ker\phi(X)$. Furthermore H_∞ is isomorphic to the group of natural transformations of the functor $T[V]$ regarded as a coalgebra.*

The self-maps given by H_∞ are those used for the splittings in section 1. This group was introduced and analyzed by the author. Subsequently, Dwyer, and Rezk showed that H_∞ exhausts all of the natural transformations of $T[V]$ which preserve the underlying coalgebra structure. The groups Lie(n) are given by the homology of certain braid groups. Namely, the braid groups arise in the next section concerning the homotopy groups of the 2-sphere in which P_n denotes the n-th pure braid group. Then $Lie(n)$ is isomrphic as a module over the symmetric group to $H_{n-1}(P_n;\mathbb{Z})$ tensored with the sign representation [6]. The groups H_n are also closely connected to low dimensional topology, and the theory of "Brunnian" links. These connections will be addressed elsewhere.

3. Braid groups, and homotopy groups of the 2-sphere

The purpose of this lecture is to outline a specific description of a group of invariant elements by describing some work of Jie Wu concerning the homotopy groups of the 2-sphere. Namely, let G be a group acting on a set S, and let S^G denote the set of fixed points under the action of G. The basic example here arises from a classical representation constructed by E. Artin in 1924 [2, 3, 5] together with recent work of Wu [31] relating this representation to the homotopy groups of the 2-sphere S^2.

Artin's representation is a homomorphism from the n-stranded braid group to the automorphism group of a free group with n generators in which the following notation is used:

1. The group B_n denotes Artin's n-stranded braid group.
2. The group P_n denotes the pure n-stranded braid group, the subgroup of B_n which leaves the endpoints of a braid unpermuted.
3. The group F_n denotes the free group on n-letters with basis $\{x_1, x_2, \cdots, x_n\}$.
4. The groups $B_n(M)$, respectively $P_n(M)$ denote the n-stranded braid group, respectively the pure the n-stranded braid group for a surface M. The n-stranded pure braid group of a surface M, $P_n(M)$, is defined as the fundamental group of the configuration space $F(M, n)$, the subspace of M^n given by $\{(m_1, m_2, \cdots, m_n) | m_i \neq m_j, i \neq j\}$. The n-stranded braid group of a surface M, written $B_n(M)$ is the fundamental group of the quotient of $F(M, n)$ by the natural action of the n-th symmetric group $F(M, n)/\Sigma_n$.

Artin's representation is given by

$$\Phi : B_n \to Aut(F_n)$$

where

1. Artin's map Φ is faithful, and
2. the automorphisms f in the image of Φ are characterized by the following 2 properties which also characterize the braid group B_n:
   (a) $f(x_1 \cdot x_2 \cdots x_n) = x_1 \cdot x_2 \cdots x_n$, and
   (b) $f(x_i) = w_i \cdot x_{\sigma(i)} \cdot w_i^{-1}$ for all i, and where σ denotes an element in the symmetric group on n-letters.

Next, consider words given by commutators in the free group F_n of the form

$$[..[y_1, y_2], y_3], \cdots], y_t]$$

where the commutator $[a, b]$ is given by $a^{-1}b^{-1}ab$, and the y_j satisfy the following two conditions:

1. All of the y_j lie in the set
$$\{x_0, x_1, x_2, \cdots, x_n\}$$
where x_0 is the product $x_1 \cdot x_2 \cdots x_n$ arising in Artin's representation, and
2. there is an equality of sets:
$$\{y_1, y_2, y_3, \cdots y_t\} = \{x_0, x_1, x_2, \cdots, x_n\}.$$

Define the group W_n to be the quotient of F_n modulo the smallest normal subgroup containing all of the words $[...[y_1, y_2], y_3], \cdots]y_t]$ as given above. Observe that by the Hall-Witt identities, the smallest normal subgroup generated by all of the words $[...[y_1, y_2], y_3] \cdots]y_t]$ is invariant under the action of B_n acting through Artin's representation. Thus there is a representation

$$\Theta : B_n \to Aut(W_n)$$

which descends from Artin's representation.

Theorem 3.1. *(J. Wu)*

1. *The group of invariant elements $W_n^{P_n}$ is the center of W_n.*
2. *The group of invariant elements $W_n^{B_n}$ is the subgroup of the center of W_n generated by all elements of order 2.*
3. *For all $n > 2$, the center of W_n is isomorphic to $\pi_{n+1}S^2$ ($= \pi_{n+1}S^3$).*

Thus Artin's representation together with classical invariants contain the seeds of the homotopy groups of the 2-sphere. The audience should be cautioned that this theorem is not useful for direct computations as is traditional in homotopy theory. The methods of proof involve simplicial groups together with the property that Artin's representation descends to an action on certain simplicial groups.

The determination of the fixed set of the action of the braid group on W_n by (combinatorial) group theoretic techniques is almost certainly beyond the reach of current methods. On the other hand, braid groups have appeared in several areas of mathematics such as group theory, homotopy theory, low dimensional topology, Galois theory, complexity of algorithms, and mathematical physics. The point is that group theoretic methods will not inform on computations, but they may admit further applications. It is the purpose of this lecture to indicate where certain structures "fit" with Wu's theorem.

Since the methods of proof are via simplicial sets, a digression concerning basic properties of simplicial sets is given now [17, 9]. First of all, a simplicial set S_* is a collection of sets S_n indexed by the non-negative integers n = 0,1,2, ... with face operations $d_i : S_n \to S_{n-1}$ with $0 \leq i \leq n$, and degeneracy operations $s_j : S_n \to S_{n+1}$ with $0 \leq j \leq n$. The face, and degeneracy operations are required to satisfy certain compatibility conditions sometimes called simplicial identities and which are described next [9, 17].

1. $d_i d_j = d_{j-1} d_i$ for $i < j$,
2. $s_i s_j = s_j s_{i-1}$ for $i > j$,
3. $$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j, \\ identity & \text{if i=j or i = j+1, and} \\ s_j d_{i-1} & \text{if } i > j+1, \end{cases}$$

An example of a simplicial set is the singular simplices of a topological space which arises in the definition of singular homology for a topological space. A second example is listed next. The simplicial circle S^1 has n-simplices S^1_n given by the set of all ordered $(n+1)$-tuples $< 0, 0, ..., 0, 1, 1, 1.., 1 >=< 0^{n-i+1}, 1^i >= x_i$ for $0 \leq i \leq n+1$.

1. The face operations
$$d_0, \cdots, d_n : S^1_n \to S^1_{n-1}$$
are specified by the following formulas.
$$d_j(x_i) = \begin{cases} x_i & \text{if } j \leq n-i, \\ x_{i-1} & \text{if } j > n-i. \end{cases}$$
2. The degeneracy operations
$$s_0, \cdots, s_n : S^1_n \to S^1_{n-1}$$
are specified by the following formulas.
$$s_j(x_i) = \begin{cases} x_i & \text{if } j \leq n-i, \\ x_{i+1} & \text{if } j > n-i. \end{cases}$$

A simplicial group G_* is a simplicial set such that

1. the simplices in degree n for every n given by G_n is a group, and
2. the face and degeneracy operations are group homomorphisms.

The homotopy groups of a simplicial group were defined by J. C. Moore [22] in a purely group theoretic way as follows:

1. Let G_* be a simplicial group.
2. Define N_q, the chains in degree q, as the intersection of the kernels of
$$d_i : G_q \to G_{q-1}$$
for $1 \leq i \leq q$,
$$N_q = \cap_{1 \leq i \leq q} ker(d_i : G_q \to G_{q-1}).$$
3. Define the group of cycles in degree q by
$$Z_q = \cap_{0 \leq i \leq q} ker(d_i : G_q \to G_{q-1}).$$
4. Define the boundaries in degree q by
$$B_q = d_0(N_{q+1}).$$

5. Then the group B_q can be shown to be a normal subgroup of Z_q, and the q-th homotopy group of G_* is defined by
$$\pi_q(G_*) = Z_q/B_q.$$

Any functor F from the category of pointed sets to the category of groups "prolongs" to a functor from the category of simplicial sets to the category of simplicial groups. Two examples of such functors are given next.

(1) the functor A which sends a pointed set X with "base-point" p to the free abelian group generated by X with the single relation that the "base-point" p is the identity element is denoted $A[X]$.

(2) the functor F which sends a pointed set X with "base-point" p to the free group generated by X with the single relation that the "base-point" p is the identity element is denoted $F[X]$.

An example is as follows. Consider the simplicial set S^1 given above. Then consider $A[S^1]$, and $F[S^1]$, the simplicial groups obtained from the functors A, and F .

It is easy to compute the homotopy groups of $A[S^1]$. They are given by $\{0\}$ in all degrees not equal to 1, and by $\mathbb{Z}$ in degree 1. (This exercise is fun, and you might try it.) The case of $F[S^1]$ turns out to contain more information.

This free group construction was developed by Milnor [21], and is discussed next where one technical condition as well as the definition of "geometric realization" is required to state the result: a simplicial set S_* is said to be "reduced" provided the set of simplices in degree 0, S_0, is a single point. In addition, there is a functor from the category of simplicial sets to the category of topological spaces given by "geometric realization" where $|S|$ denotes the geometric realization of a simplicial set S. The realization is defined by
$$|S| = \cup_{q\geq 0}\Delta[q] \times S_q/R$$
where $\Delta[q]$ denotes the q-simplex, and "R" is the equivalence relation generated by

1. $(v, d_i x)$ is equivalent to $(\epsilon_i(v), x)$ for v in $\Delta[q-1]$, x in S_q, and with $\epsilon_i : \Delta[q-1] \to \Delta[q]$ given by the inclusion of the i-th face, and
2. $(v, s_i x)$ is equivalent to $(\eta_i(v), x)$ for v in $\Delta[q+1]$, x in S_q, and with $\eta_i : \Delta[q+1] \to \Delta[q]$ given by the projection to the i-th face.

Theorem 3.2. *(Milnor) Let K be a reduced simplicial set. Then the geometric realization of $F[K]$ is homotopy equivalent to $\Omega\Sigma|K|$.*

One corollary is the starting point of Wu's investigation.

Corollary 3.3. *There is an isomorphism of groups*
$$\pi_q F[K] \to \pi_q\Omega\Sigma|K|.$$
Thus $\pi_q F[S^1]$ is isomorphic to $\pi_q\Omega S^2 \cong \pi_{q+1}S^2$

The point of this corollary is that one can view the homotopy groups of the 2-sphere as a combinatorially defined object which can be studied through combinatorial methods. These methods are frequently quite interesting, although they rarely have immediate computational value. Part of the features of the structure here is the next theorem which indicates part of the role of the center in simplicial groups.

Theorem 3.4. *If G is a reduced simplicial group, then $\pi_n G$ is contained in the center of the quotient group G_n modulo B_n.*

Wu then defines an "r-centerless" simplicial group G_* which is a simplicial group for which the center of G_n is trivial for $n \geq r$. An example of such a G arises in case G_n is a free group on at least 2 generators for $n \geq r$. A specific example is given by $F[S^1]$ which is "2-centerless". In degree 1, the simplicial group $F[S^1]$ is isomorphic to the integers, and is thus not "1-centerless".

Theorem 3.5. *If G is a reduced "r-centerless" simplicial group, then $\pi_n G$ for $n \geq r+1$ is equal to the center of G_n/B_n.*

Wu then applies this to $F[S^1]$ in order to prove his theorem on fixed points, and the homotopy groups of the 2-sphere. There are further connections between this problem, and other features of braid groups.

There is another simplicial group AP_* which in degree n is Artin's pure braid group on $n+1$ strands, and which gives some further information concerning Wu's theorem. Thus this simplicial group is isomorphic to the integers in degree 1. In joint work of Wu, and the author, the unique homomorphism of simplicial groups

$$\Theta : F[S^1] \to AP_*$$

which sends a generator in degree one to a generator is studied. Some properties are listed next:

1. $\Theta : F[S^1] \to AP_*$ is a monomorphism in each degree,
2. thus the q-th homotopy group of $F[S^1]$ is a subquotient of P_{q+1},
3. the quotient simplicial set $AP_*/F[S^1]$ has geometric realization which is homotopy equivalent to the 2-sphere, and
4. the (simplicial) loop space of AP_* is isomorphic to Milnor's free group construction $F[\Delta[1]]$ where $\Delta[1]$ is the simplicial one simplex.

The morphism of simplicial groups $\Theta : F[S^1] \to AP_*$ is related to the set of isotopy classes of n-component links, $\mathcal{L}_n$, as follows. There is a morphism of sets from the n-th pure braid group to the set of isotopy classes of n-component links as given in classical work of Alexander, and Markov [5]:

$$AP_n \to \mathcal{L}_n,$$

and

$$AP_* \to \cup_{n \geq 0} \mathcal{L}_n.$$

One question which arises in this context is as follows. Describe the image of $F[S^1]$, as well as the subgroups given by chains, cycles, and boundaries in the set of isotopy classes of links. Two examples are given next.

1. A cycle which represents the Hopf map $\eta : S^3 \to S^2$ is given by $[x_1, x_2]$ in the set of 2-simplices for $F[S^1]$. The image of this cycle in the set of isotopy classes of 3 component links is the Borromean rings.
2. The cycle $[x_1, x_2]^2$ represents the Whitehead product $[\iota_2, \iota_2]$. The image of this cycle is a 3-component link where two circles are "twisted around each other twice" while the third circle links the other 2 as if they were the Borromean rings. (Try it. It is interesting.)

Two important questions which arise in Wu's work are given next. Consider the n natural homomorphisms

$$d_i : P_n \to P_{n-1}$$

obtained by deleting the $i-th$ strand for $1 \leq i \leq n$. Thus there is an induced homomorphism

$$d : P_n \to \prod_{1 \leq i \leq n} P_{n-1}.$$

The kernel of this map is a free group. What is a basis for the kernel of this map ?

A second homomorphism is given by

$$\gamma : P_n \to P_n(S^2),$$

the natural quotient of the pure braid group for the plane to the pure braid group for the 2-sphere S^2. This homomorphism is obtained by applying the fundamental group to the natural inclusion of the configuration spaces $F(\mathbb{R}^2, n) \to F(S^2, n)$.

What is the kernel of the natural homomorphism

$$\gamma \times d : P_n \to P_n(S^2) \times \prod_{1 \leq i \leq n} P_{n-1}?$$

4. Cohomology of symmetric groups, and other groups

This section addresses classical work on the cohomology of the symmetric groups, certain subgroups of symmetric groups, and related groups. A smattering of information about problems, and applications is included. First recall the ingredients required for the definition of the homology, and the cohomology of a discrete group π.

1. An abelian group A is said to be a trivial $\mathbb{Z}[\pi]$-module, or a trivial π-module, provided A is a module over the integral group ring of π, $\mathbb{Z}[\pi]$, such $\sigma(a) = a$ for every element a in A, and every element σ in π.
2. Let $\mathbb{Z}$ be a trivial $\mathbb{Z}[\pi]$-module, let M be a left $\mathbb{Z}[\pi]$-module, and let N be a right $\mathbb{Z}[\pi]$-module.
3. Let
$$\cdots \to R_3 \to R_2 \to R_1 \to R_0 \to \mathbb{Z} \to \{0\}$$
be a free resolution of $\mathbb{Z}$ by free left $\mathbb{Z}[\pi]$-modules R_i.

The definitions of group homology, and cohomology are as follows.

1. The homology of π with N coefficients is
$$H_*(\pi; N) = Tor_*^{\mathbb{Z}[\pi]}(\mathbb{Z}, N),$$

and is the homology of the chain complex

$$\cdots \to N \otimes_{\mathbb{Z}[\pi]} R_3 \to N \otimes_{\mathbb{Z}[\pi]} R_2 \to N \otimes_{\mathbb{Z}[\pi]} R_1 \to N \otimes_{\mathbb{Z}[\pi]} R_0.$$

2. The cohomology of π with M coefficients is

$$H^*(\pi; M) = Ext^*_{\mathbb{Z}[\pi]}(\mathbb{Z}, M),$$

and is the cohomology of the cochain complex

$$M^0 \to M^1 \to \cdots \to M^i \to M^{i+1} \to M^{i+2} \to \cdots$$

where $M^i = Hom_{\mathbb{Z}[\pi]}(R_i, M)$.

These functors are frequently informative, and frequently computable. They provide useful ways of measuring interesting behavior. The first few classical results are as follows.

Theorem 4.1. *Let π be a finite group of order n.*

1. *Then $n \cdot H^j(\pi; M) = 0$, and $n \cdot H_j(\pi; N) = 0$ for all $j > 0$. Thus if $j > 0$, $H^j(\pi; M)$ respectively $H_j(\pi; N)$ is the direct sum of its p-primary components ${}_pH^j(\pi; M)$ respectively ${}_pH_j(\pi; N)$ for all primes p which divide n.*
2. *Let π_p denote the p-Sylow subgroup π. Then the restriction map*

$$_pH^*(\pi; M) \to H^*(\pi_p; M)$$

is a split monomorphism.

3. *If π is a finite group with abelian p-Sylow subgroup π_p, then the mod-p cohomology of π is given by $H^*(\pi_p, \mathbb{Z})^{N(\pi_p)}$ the invariant elements under the action of the normalizer $N(\pi_p)$ of π_p in π.*

Again, the elementary abelian groups are basic examples, as we have seen in several of the lectures. Their cohomology is classical, basic, and important.

Theorem 4.2.
1. *The cohomology ring $H^*((\mathbb{Z}/2\mathbb{Z})^n; \mathbb{F}_2)$ is a polynomial ring with generators $x_1, ..., x_n$ of degree 1.*
2. *If $r \geq 2$ or p is an odd prime, then the cohomology ring $H^*((\mathbb{Z}/p^r\mathbb{Z})^n; \mathbb{F}_p)$ is the tensor product of an exterior algebra with generators $x_1, ..., x_n$ of degree 1 tensored with a polynomial ring with generators $y_1, ..., y_n$ of degree 2. Furthermore, the $r-th$ Bockstein β_r of x_i is defined and satisfies the formula $\beta_r(x_i) = y_i$.*

A second important example is the symmetric group on n letters Σ_n. The homology, and cohomology of symmetric groups is addressed next. Let Σ_∞ denote the colimit of the Σ_n under the natural inclusion. There are analogues for Artin's braid groups $Br_n \to Br_{n+1}$ with colimit denoted Br_∞. The homology of these groups is related to the homology of certain useful topological spaces.

One connnection between the cohomology of the symmetric groups, and that of elementary abelian p-groups is as follows: The regular representation of $(\mathbb{Z}/2\mathbb{Z})^n$, a homomorphism $(\mathbb{Z}/2\mathbb{Z})^n \to \Sigma_{2^n}$, induces a map $H^*(\Sigma_{2^n}; \mathbb{F}_2) \to H^*((\mathbb{Z}/2\mathbb{Z}))^n; \mathbb{F}_2)$ which has image given by the Dickson algebra on n generators, the invariant subalgebra under the natural $Gl(n, \mathbb{F}_2)$-action. There is a further connection to spaces of continuous functions as described next.

Consider the natural suspension map $X \to \Omega\Sigma X$ which is the adjoint of the identity $\Sigma X \to \Sigma X$. Iterating, there is a map $\Omega^n\Sigma^n X \to \Omega^{n+1}\Sigma^{n+1}X$ with $QX = \varinjlim \Omega^n\Sigma^n X$. Write $\Omega_0^n\Sigma^n X$, and Q_0X for the respective path component of the identity. One feature of the spaces QX is that if X is a CW-complex, then the i-th homotopy group of QX is isomorphic to the i-th stable homotopy group of X. Thus properties of QX impact the stable homotopy groups of X.

The following theorem concerning the symmetric groups has input from many people including Araki, Kudo, Nakaoka, Dyer, Lashof, Barratt, Priddy, and Quillen [4, 1, 24, 10, 15]. The second part was proven in work of May, Segal, and the author [29, 6, 18].

Theorem 4.3. *Assume that homology is taken with any trivial coefficients. (Namely, the fundamental group of each space acts on the coefficients by the identity map.)*

1. *There is a map $B\Sigma_\infty \to Q_0S^0$ which induces a homology isomorphism.*
2. *There is a map $BBr_\infty \to \Omega_0^2S^2$ which induces a homology isomorphism.*

A more general version is the Kan-Thurston theorem: If X is a path-connected CW-complex, then there exists a $K(\pi,1)$ together with a map $K(\pi,1) \to X$ which induces a homology isomorphism with any trivial coefficients. Thus, on the level of homology, many reasonable spaces behave as if they are $K(\pi,1)'s$.

Some preparation for applications of the homological properties above are given next. There are maps $\mathbb{RP}^\infty \to \Omega_0^\infty S^\infty$ which induce an isomorphism on the level of fundamental groups given by $\mathbb{Z}/2\mathbb{Z}$. One such map is induced by reflection through the hyperplane orthogonal to a given line through the origin. This gives a map to the -1-component of $O(n)$, $\mathbb{RP}^{n-1} \to O(n)$. Translating to the $+1$-component of $O(n)$, and letting n go to infinity gives $P^\infty \to SO \to \Omega_0^\infty S^\infty$. Let $\Theta : Q\mathbb{RP}^\infty \to \Omega_0^\infty S^\infty$ denote any extension which is a loop map. The following is the 2-primary Kahn-Priddy theorem. There is an odd primary version.

Theorem 4.4. *The map Θ has a 2-local section. Thus after localization at $p = 2$, the following is satisfied:*

1. *There is a 2-local homotopy equivalence*
$$Q\mathbb{RP}^\infty \to \Omega_0^\infty S^\infty \times X$$
for some space X.
2. *The map $\Theta : Q\mathbb{RP}^\infty \to \Omega_0^\infty S^\infty$ gives a split epimorphism on the 2-primary components of homotopy groups.*

A non-stable analogue of this theorem is gotten as follows. Consider the cofibration $S^2 \to S^2 \to P^3(2)$ where the map $S^2 \to S^2$ is degree 2. Applying the pointed mapping functor to this cofibration gives a fibration $q : \Omega^2S^n \to \Omega^2S^n$ given by the H-space squaring map with homotopy theoretic fibre denoted $map_*(P^3(2), S^n)$, where $map_*(A, B)$ denotes the space of pointed maps from A to B. Notice that $P^3(2)$ is homotopy equivalent to the suspension of the projective plane $\Sigma\mathbb{RP}^2$. Let W_n denote the homotopy theoretic fibre of the double suspension $E^2 : S^{2n-1} \to \Omega^2S^{2n+1}$.

Theorem 4.5. 1. *If p is an odd prime, then there is p-local equivalence*
$$map_*(P^3(p), S^{2p+1}) \to \Omega^2S^3 < 3 > \times W_p.$$

Thus $map_*(P^3(p), S^{2p+1}) \to \Omega^2 S^3 < 3 >$ *induces a split epimorphism on the p-primary component of homotopy groups, and p annihilates the p-primary component of* $\pi_i S^3$ *for any* $i > 3$.

2. *There is a 2-local equivalence*

$$map_*(\Sigma\mathbb{RP}^2, S^5) \to \Omega^2 S^3 < 3 > \times W_2.$$

Thus $map_*(\Sigma\mathbb{RP}^2, S^5) \to \Omega^2 S^3 < 3 >$ *induces a split epimorphism on the 2-primary component of homotopy groups, and 4 annihilates the 2-primary component of* $\pi_i S^3$ *for any* $i > 3$.

The method of proofs of these theorems is given by constructing maps, and then appealing to the cohomological results in Theorem 4.2 above to prove that a map is an equivalence. The Kahn-Priddy theorem is, of course, due to D. S. Kahn, and S. Priddy. The odd primary result in Thorem 4.5 above is due to P. Selick while the 2-primary theorem is due to the author.

Next consider further extensions given by the wreath product $\Sigma_n \wr G$ which is the group extension

$$1 \to G^n \to \Sigma_n \wr G \to \Sigma_n \to 1$$

which is specified as follows:

1. As a set $\Sigma_n \wr G$ is isomorphic to $\Sigma_n \times G^n$ with elements written as $(\sigma; g_1, ..., g_n)$.
2. The multiplication is specified by
$$(\sigma; g_1, ..., g_n)(\tau; h_1, ..., h_n) = (\sigma\tau; g_{\tau^{-1}(1)}h_1, ..., g_{\tau^{-1}(n)}h_n).$$
3. The fundamental group of $E\Sigma_n \times_{\Sigma_n} X^n$ is isomorphic to $\Sigma_n \wr G$ for path-connected spaces X with $\pi_1(X) = G$.

There is a slightly more general, and useful definition of the wreath product which fits in other contexts. Namely, given any homomorphism $f : \Pi \to \Sigma_n$, define the wreath product $\Pi \wr G$ (where the notation does not display the dependence of the extension on the homomorphism f) as a pull-back:

$$\begin{array}{ccc} \Pi \wr G & \longrightarrow & \Pi \\ \downarrow & & \downarrow \\ \Sigma_n \wr G & \longrightarrow & \Sigma_n \end{array}$$

Thus there is a morphism of group extensions:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G^n & \longrightarrow & \Pi \wr G & \longrightarrow & \Pi & \longrightarrow & 1 \\ {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & \downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle 1}\downarrow \\ 1 & \longrightarrow & G^n & \longrightarrow & \Sigma_n \wr G & \longrightarrow & \Sigma_n & \longrightarrow & 1 \end{array}$$

Consider the Lyndon-Hochschild-Serre spectral sequence for this extension with coefficients in a field $\mathbb{F}$ in homology. Then,

$$E^2_{s,t} = H_s(\Pi; H_t(G^n, \mathbb{F})).$$

A modification and interpretation of some earlier results of Steenrod are given in [6], Lemmas 4.1-4.3, and these imply that $E^2 = E^\infty$. A similar assertion applies in cohomology if $H^*(G;\mathbb{F})$ is of finite type.

Next, notice that $H_*(G^n;\mathbb{F})$ is isomorphic to $V^{\otimes n}$ for $V = H_*(G;\mathbb{F})$. As a module over Σ_n, and hence as a module over G, $V^{\otimes n}$ is a direct sum of cyclic Σ_n-modules which depend on choices of partitions of $\{1,2,...,n\}$, say M_Λ. For example, if G is the trivial group, then $V = \mathbb{F}$ concentrated in degree 0, and the modules M_Λ are always trivial.

As a second example, assume that the b_α run over a totally ordered basis for $H_*(G;\mathbb{F})$. Consider the cyclic Σ_n-module generated by the element

$$\lambda = b_{\alpha_1}^{\otimes n_1} \otimes b_{\alpha_2}^{\otimes n_2} \cdots \otimes b_{\alpha_q}^{\otimes n_q}$$

where

1. P is an ordered partition of n such that $P = (n_1, n_2, \cdots, n_q)$ for $n_i > 0$ with $n_1 + n_2 + \cdots + n_q = n$,
2. B is a sequence of strictly increasing basis elements with $B = (b_{\alpha_1}, b_{\alpha_2}, \cdots, b_{\alpha_q})$ for $b_{\alpha_1} < b_{\alpha_2} < \cdots < b_{\alpha_q}$, and
3. the pairs labelled by $\Lambda = (P,B)$ run over all distinct such pairs (P,B).

Then a direct sum decomposition for the Π-module $V^{\otimes n}$ is given by

$$\oplus_{\Lambda=(P,B)} M_\Lambda.$$

Consequently, there is a homology isomorphism

$$H_*(\Pi \wr G;\mathbb{F}) \to \oplus_\Lambda H_*(\Pi; M_\Lambda).$$

Furthermore, if $\Pi = \Sigma_n$, and $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, then $H_*(\Sigma_n; M_\Lambda)$ is isomorphic, apart from a degree shift given by the degree of $b_{\alpha_1}^{\otimes n_1} \otimes b_{\alpha_2}^{\otimes n_2} \cdots \otimes b_{\alpha_q}^{\otimes n_q}$, to

$$H_*(\Sigma_{n_1} \times \Sigma_{n_2} \times \cdots \Sigma_{n_q}; \mathbb{Z}/2\mathbb{Z})$$

where $\Sigma_{n_1} \times \Sigma_{n_2} \times \cdots \Sigma_{n_q}$ is the subgroup of Σ_n that fixes $b_{\alpha_1}^{\otimes n_1} \otimes b_{\alpha_2}^{\otimes n_2} \cdots \otimes b_{\alpha_q}^{\otimes n_q}$. Thus the mod-2 homology of the wreath product $\Sigma_n \wr G$ is given in terms of the mod-2 homology of subgroups of Σ_n with trivial coefficients in $\mathbb{F}_2$.

There is an analogous description over a field of odd characteristic for which modifications using trivial coefficients or coefficients in the sign representation are used. This case will not be addressed in these abbreviated notes.

Next notice that the Lyndon-Hochschild-Serre spectral sequence in cohomology with trivial coefficients in $\mathbb{F}$ collapses for the extension

$$1 \to G^n \to \Pi \wr G \to \Pi \to 1$$

by a cochain level argument provided the cohomology of G with $\mathbb{F}$ coefficients is of finite type. The resulting E_2-term is dually given in terms of (1) the cohomology of G, and (2) ordered partitions of n. This remark is stated as the following theorem.

Theorem 4.6. *The homology of $\Pi \wr G$ with field coefficients $\mathbb{F}$ is given by*

$$H_*(\Pi; H_*(G^n;\mathbb{F})) = \oplus_\Lambda H_*(\Pi; M_\Lambda).$$

Furthermore, the homology is naturally bigraded by

$$H_s(\Pi; H_t(G^n; \mathbb{F}))$$

in bidgree (s,t)*. The homology in total degree* q *is given by*

$$H_q(\Pi \wr G; \mathbb{F}) = \oplus_{s+t=q} H_s(\Pi; H_t(G^n; \mathbb{F})).$$

In case $G = \Sigma_n$*, these homology groups are given in terms of*

1. *the additive structure of* $H_*(G; \mathbb{F})$*, and*
2. *ordered partitions of* n *as described above.*

A second interpretation of E_2 fits with the subjects in this conference. This interpretation will be illustrated in several cases below.

1. Assume that $G = \mathbb{Z}/2\mathbb{Z}$, and that the coefficient field is $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. Then $H^*(BG^n; \mathbb{F}_2)$ is a polynomial ring in n indeterminates of degree 1 with the Σ_n-action specified by the polynomials in the fundamental representation of Σ_n, $\mathbb{F}_2[V_n]$. Thus the $E_2^{s,t}$-term of the Lyndon-Hochschild-Serre spectral sequence abutting to the mod-2 cohomology of $\Sigma_n \wr G$ is given by

   $$H^s(\Sigma_n; H^t((\mathbb{Z}/2\mathbb{Z})^n; \mathbb{F}_2).$$

   By the above remarks, this spectral sequence collapses, and so $E_2 = E_\infty$. Hence $H^*(\Sigma_n; \mathbb{F}_2[V_n])$ is given in terms of the cohomology of subgroups and partitions listed above.

   In addition, the cohomology of $\Sigma_n \wr G$ with field coefficients $\mathbb{F}$ is naturally bigraded, and is given by $H^s(\Sigma_n; H^t((G)^n; \mathbb{F})$ in bidegree (s,t). The invariant subalgebra $H^*(BG^n; \mathbb{F}_2)^{\Sigma_n}$ is given by $H^0(\Sigma_n; \mathbb{F}_2[V_n])$. The ring of invariants is precisely the mod-2 Dickson algebra, and has bidgree $(0,*)$ where $*$ denotes the standard grading for the Dickson algebra.

2. A similar assertion follows for $G = S^1$, and where the coefficient field is $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. The wreath product construction here is given by the normalizer of the maximal torus in the unitary group $U(n)$. Similarly, $H^*(BG^n; \mathbb{F}_p)$ is a polynomial ring in n indeterminates of degree 2 with the Σ_n-action specified by the fundamental representation of Σ_n, $\mathbb{F}_p[V_n])$. Again, $H^*(\Sigma_n; \mathbb{F}_p[V_n])$ is given in terms of the cohomology of subgroups of the symmetric group with the trivial representation, and partitions listed above. The resulting answer is (1) the cohomology of the wreath product, and (2) identifies the ring of invariants as the summand of cohomology group of the wreath product concentrated in bidegrees $(0,*)$.

3. Let $G = \mathbb{Z}/p^r\mathbb{Z}$, with the coefficient field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for an odd prime p. Then $H^*(BG^n; \mathbb{F}_p)$ is the tensor product of a polynomial ring in n indeterminates of degree 2 with an exterior algebra in n indeterminates of degree 1, $E[V_n]$. The Σ_n-action is specified by that on the fundamental representation which is extended multiplicatively to $\mathbb{F}_p[V_n] \otimes E[V_n]$. Again, $H^*(\Sigma_n; \mathbb{F}_p[V_n] \otimes E[V_n])$ is given in terms of the cohomology of subgroups of Σ_n with coefficients in either the trivial representation or the sign representation. The subgroups are of the form $\Sigma_{n_1} \times \Sigma_{n_2} \times \cdots \Sigma_{n_q}$ and fix $b_{\alpha_1}^{\otimes n_1} \otimes b_{\alpha_2}^{\otimes n_2} \cdots \otimes b_{\alpha_q}^{\otimes n_q}$ up to a sign.

4. These constructions are useful in characteristic zero where symmetric groups are replaced by other discrete groups such as $SL(2,\mathbb{Z})$, $Sp(2g,\mathbb{Z})$, mapping class

groups, or braid groups. In these cases, some answers are given in terms of constructions in analytic number theory. (Please see the problems below.)

5. Problems

(1): Let G be a discrete group together with a representation $\rho : G \to GL(n, R)$ for R either a finite field or the integers. Write V_n for the direct sum of n copies of R, the fundamental represention of $GL(n, R)$.

Functors given by $P[V_n]$, $E[V_n]$, and $T[V_n]$, the polynomial ring, exterior algebra, and tensor algebra respectively generated by V_n are naturally $GL(n, R)$-modules. What can be said about the cohomology groups of G with coefficients taken in $P[V_n]$, $E[V_n]$, and $T[V_n]$?

The motivation for this question is that there have been useful applications of known examples as suggested below.

1. When G is the symmetric group on n letters, Σ_n, with the natural n-dimensional representation, then the cohomology with coefficients in $P[V_n]$, $E[V_n]$, or their tensor product is known implicitly from work of Steenrod and others. For example, over the field of 2 elements, $H^*(\Sigma_n; \mathbb{F}_2[V_n])$ gives the $E_2 = E_\infty$ term of the Lyndon-Hochschild-Serre spectral sequence abutting to the mod-2 cohomology of the wreath product
$$\Sigma_n \wr \Sigma_2$$
(where $\Sigma_n \wr G$ is a group extension $1 \to G^n \to \Sigma_n \wr G \to \Sigma_n \to 1$).

   The additive structure of the E_2- term is given in terms of partitions, and the cohomology of certain choices of subgroups of the symmetric groups obtained from these partitions.

   Related remarks concerning representations, as well as subgroups of the symmetric groups are recalled in Cohen's lecture notes.

2. When $G = SL(2, \mathbb{Z})$, and $V_2 = \mathbb{Z} \oplus \mathbb{Z}$, the rational cohomology of G with coefficients in $P[V_2]$, $E[V_2]$, or their tensor product is known in terms of classical modular cusp forms based on the standard $SL(2, \mathbb{Z})$-action on the upper 1/2-plane. These calculations trace back to work of Eichler, and Shimura on automorphic forms [11, 29, 13].

3. When G is $GL(n, R)$, then S. Betley has proven a general vanishing theorem for these coefficients as n goes to infinity. According to Betley, the analogous question for the symplectic groups remain undecided.

4. When G is the mapping class group for a closed surface of genus g, Γ_g, there is an epimorphism
$$\Gamma_g \to Sp(2g, \mathbb{Z}).$$
The cohomology groups $H^*(\Gamma_g; \mathbb{Q}[V_{2g}])$ have been studied by E. Looijenga who has obtained a stability result. These groups as well as $H^*(\Gamma_g; \mathbb{Q}[V_{2g}] \otimes E[V_{2g}])$ inform on the cohomology of mapping class groups for punctured surfaces.

5. In the special case of $R = \mathbb{Z}/2\mathbb{Z}$, there is an isomorphism $\Sigma_6 \to Sp(4, R)$. What is $H^*(\Sigma_6; R[V_4] \otimes_{\mathbb{Z}/2\mathbb{Z}} E[V_4])$?

(2): Describe the Dyer-Lashof algebra as a collection of natural transformations as suggested by Bisson's lecture, and in the spirit of Smith's lectures. Selick, and Campbell have given a construction which pieces together the Steenrod algebra together with the Dyer-Lashof algebra into one giant natural algebraic construction. Is this object describing the natural transformations of certain natural choices of functors ?

(3): Two important questions which arise in Wu's work and which were stated above are as follows: Consider the n natural homomorphisms

$$d_i : P_n \to P_{n-1}$$

obtained by deleting the $i-th$ strand for $1 \leq i \leq n$. What is a free basis for the kernel of the induced homomorphism

$$d : P_n \to \prod_{1 \leq i \leq n} P_{n-1}?$$

What is the kernel of the natural homomorphism

$$\gamma \times d : P_n \to P_n(S^2) \times \prod_{1 \leq i \leq n} P_{n-1}?$$

(4): Characterize the image of $F[S^1]$, as well as the subgroups given by chains, cycles, and boundaries in the set of isotopy classes of links.

(5): Find useful group theoretic characterizations of the homotopy groups of spheres. Do these "fit" with the braid groups ? How does the structure of the isotopy classes of n-component links impact the homotopy groups of the 2-sphere ? Find interesting analogues of the Kahn-Priddy theorem which apply to the $(2n+1)$-sphere and which are natural extensions of the analogue for the 3-sphere.

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